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Let's consider how S-matrix elements $\langle out | in \rangle$ transform under Poincaré

— Translations —

(1) $$\langle out | in \rangle = \langle out | e^{iP\cdot a} e^{-iP\cdot a} | in \rangle = e^{i(\sum_{out} p - \sum_{in} k)} \langle out | in \rangle$$

$\langle out| = {}_{out}\langle p_1^{h_1} \ldots p_m^{h_m}|$, $\quad |in\rangle = |k_1^{h_1} \ldots k_n^{h_n}\rangle_{in}$, $\quad P$ conserved

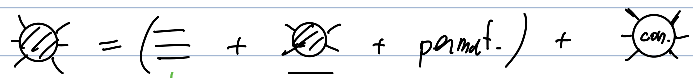
(2) $$\boxed{\sum_{out} p = \sum_{in} k}$$

Removing the disconnected subamplitudes, where momentum is conserved separately in each cluster

(3) $$S(k \to p)\big|_{conn} = (2\pi)^4 \delta^4\left(\sum_{out} p - \sum_{in} k\right) \mathcal{M}(k_1^{h_1} \ldots k_n^{h_n} \to p_1^{\bar h_1} \ldots p_m^{\bar h_m})$$

the (connected) scattering amplitude

[[as opposed to sum of product of disconnected terms that produce product of momentum-δ^4 conservation, e.g.



$\longrightarrow \propto (2\pi)^4 \delta^4(\sum_{in} k_i - \sum_{out} p_i)$

$\propto \prod_i \left(\delta^4(k_i - p_i)\, \delta_{h_i}^{\bar h_i}\, \delta_{j_i}^{\bar j_i}\right) \quad \propto \left[\delta^4(k_3 - p_3)\, \delta_{h_3}^{\bar h_3}\, \delta_{j_3}^{\bar j_3}\right]\left[\prod_{i\neq 3} \delta^4(k_i - p_i)\, \delta_{h_i}^{\bar h_i}\, \delta_{j_i}^{\bar j_i}\right]$ (+ permutations)]]

— Lorentz —

(4) $${}_{out}\langle p_1^{\bar h_1} \ldots p_m^{\bar h_m} | k_1^{h_1} \ldots k_n^{h_n} \rangle_{in} = {}_{out}\langle p_1^{\bar h_1} \ldots p_m^{\bar h_m} | U^\dagger(\Lambda)\, U(\Lambda) | k_1^{h_1} \ldots k_1^{h_n} \rangle_{in}$$

$$= {}_{out}\langle (\Lambda p_1)^{\bar h_1'} \ldots (\Lambda p_m)^{\bar h_m'} | (\Lambda k_1)^{h_1'} \ldots (\Lambda k_n)^{h_n'} \rangle_{in}\, D^{(j_1)}_{h_1' h_1}(W(\Lambda, k_1)) \ldots D^{(j_n)}_{h_n' h_n}(W(\Lambda, k_n))\, D^{(\bar j_1)}_{\bar h_1' \bar h_1}(W(\Lambda, p_1))^* \ldots D^{(\bar j_m)}_{\bar h_m' \bar h_m}(W(\Lambda, p_m))^*$$

j-dim irrep of Little Group; Wigner rotation

and, since the $\delta^4(\sum k_i - \sum p_j)$ in each partition of S are all Lorentz invariant, and $\delta_{h_i}^{\bar h_i}$ is Little-Group invariant, then the (connected) scattering amplitude transforms as following

$$(5)\quad \boxed{\mathcal{M}(k_1^{h_1}\dots k_n^{h_n} \to p_1^{\bar h_1}\dots p_m^{\bar h_m}) = \mathcal{M}((\Lambda k_1)^{h_1'}\dots(\Lambda k_n)^{h_n'} \to (\Lambda p_1)^{\bar h_1'}\dots(\Lambda p_m)^{\bar h_m'})\, D^{(j_1)\,h_1}_{h_1'}(W_1)\dots D^{(j_n)\,h_n}_{h_n'}(W_n)\, D^{(\bar j_1)\,\bar h_1 *}_{\bar h_1'}(\bar W_1)\dots}$$

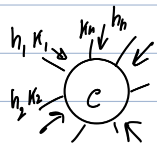
For massless particles, in particular, $D_{h'}^{\ h} = \delta_{h'}^{\ h}\exp(-i\theta_W(\Lambda,k)\,h)$ so that (5) reads

$$(6)\quad \boxed{\mathcal{M}(\{k_i^{h_i}\} \to \{p_j^{\bar h_j}\}) = \mathcal{M}(\{\Lambda k_i^{h_i}\} \to \{\Lambda p_j^{\bar h_j}\})\, e^{-ih_1\theta_1}\dots e^{-ih_n\theta_n}\, e^{i\bar h_1\bar\theta_1}\dots e^{i\bar h_m\bar\theta_m}}$$

massless-only particles

Remark: the outgoing states transform exactly like ingoing states with opposite helicity so that's convenient to adopt the convention that all particles are treated as if they were all incoming, just flipping the helicity of the outgoing ones. Likewise, it's convenient to redefine their momenta too $p_i \to -k_i$ so that everything is symmetric ($\delta^4(\sum_i k_i)$ - of conservation, LSZ, ...), and the sign of k_i^0 telling of which physical process one is referring. If internal charges are present they are flipped too:

$$(7)\qquad \mathcal{M}_{h_1\dots h_n}(k_1,\dots,k_n)\,(2\pi)^4\delta^4\Big(\sum_{i=1}^{n} k_i\Big) =$$



all incoming convention
(n total number of legs)

(the only caveat is actually a phase factor, that can be fixed using CPT that sends $|in\rangle \leftrightarrow |out\rangle$ $CPT|k^h\rangle = (-1)^h|k^{-h}\rangle$ so that one actually includes a further $(-1)^h$ when flipping from in $\leftrightarrow$ out)

Provisional Lesson:

The helicities h_i are (Poincaré-invariant) labels of massless amplitudes that are functions of momenta, which need to satisfy

$$(8)\quad \boxed{\mathcal{M}_{h_1\dots h_n}(k_1\dots k_n) = \mathcal{M}_{h_1\dots h_n}(\Lambda k_1\dots\Lambda k_n)\, e^{-ih_1\theta_1}\dots e^{-ih_n\theta_n}}$$

($\theta_i = \theta(\Lambda, k_i)$ Wigner rotation)

Remark:

The LSZ-way of calculating scattering amplitudes automically solves the (5) and (6), because of the way polarizations transform. For example, a massive vector gives:

$$(9)\quad \mathcal{M}(k^{h}\ldots) = LSZ \cdot \langle A_\mu(x)\ldots\rangle = LSZ_{\text{rest}}\cdot\left(\frac{\Box+m^2}{iZ}\right)\int d^4x\, e^{ikx}\,\varepsilon^{h}_\mu(k)\,\langle A_\mu(x)\ldots\rangle$$

$$= LSZ_{\text{rest}}\cdot\left(\frac{\Box+m^2}{iZ}\right)\int d^4x\, e^{i\Lambda k\cdot x}\,\varepsilon^{h}_\mu(k)\,\langle A^\mu(\Lambda^{-1}x)\ldots\rangle \qquad \updownarrow \text{ since } U|0\rangle = |0\rangle$$

$$= LSZ|_{\text{rest}}\,\frac{\Box+m^2}{iZ}\int d^4x\, e^{i\Lambda k\cdot x}\,\underbrace{\varepsilon^{h}_\mu(k)\,\Lambda_\nu{}^{\mu}}_{\varepsilon^{h'}_\nu(\Lambda k)\,D^{(3)h}_{h'}(W)}\,\langle A_\nu(x)\ldots\rangle$$

$$= \mathcal{M}(\Lambda k^{h'}\ldots)\, D^{(3)h}_{h'}(W)\ldots$$

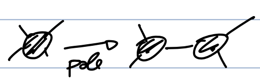
On the other hand, for a massless vector $\varepsilon^{h}_\mu(k) \to e^{-ih\theta}\left(\varepsilon^{h}_\mu(\Lambda k) - k_\mu \#\right)$, so that (6) or (8) are ok as long as $\mathcal{M}_\mu$ defined by $\mathcal{M} = \varepsilon^\mu \mathcal{M}_\mu$ is transverse

$$(10)\qquad \boxed{k_\mu \mathcal{M}^\mu = 0} \qquad \text{(Ward identity)}$$

In other words, we have a well-defined algorithm to produce amplitudes that respect Poincaré if we start with a Lagrangian, calculate correlators (e.g. via Path integral or Feynman) and finally LSZ-project on the singularities.

— On-shell Methods —

Let's change now perspective: is Poincaré + unitarity enough to actually guess the right answer directly for $\mathcal{M}$? The answer, for massless particles especially, is yes for on-shell 3pt-functions , out of which one can build higher-n-points via unitarity/factoriz.



. Let's restrict for concreteness to massless particles only, $k_i^2 = 0$, but allow ourself complex k_μ (after all in LSZ the Fourier transf. can be analytically extended)

Q: What functions $M_{h_1\dots h_n}(K_1\dots K_n)$ satisfy Eq. (8) ?

This seems hard, but in the right variables — spinors — become trivial!

Let's remind that

(11) $$K^\mu = \frac{1}{2}\bar\sigma^{\mu\dot\alpha\alpha}K_{\alpha\dot\alpha} \longleftrightarrow \begin{cases} K_{\alpha\dot\alpha} = K_\mu \sigma^\mu_{\alpha\dot\alpha} \\ K^{\dot\alpha\alpha} = K_\mu \bar\sigma^{\mu\dot\alpha\alpha}\end{cases}$$

$(\mathrm{Tr}(\bar\sigma^\mu\sigma^\nu) = 2\eta^{\mu\nu}$ from Clifford Alg. $(\bar\sigma^\mu\sigma^\nu)^{\dot\beta}_{\dot\alpha} = 2\eta^{\mu\nu}\delta^{\dot\beta}_{\dot\alpha}$ $\bar\sigma^{\mu\dot\alpha\alpha} \equiv \varepsilon^{\alpha\beta}\varepsilon^{\dot\alpha\dot\beta}\sigma^\mu_{\beta\dot\beta}$ $)$

so that $\det K_{\alpha\dot\alpha} = K^2 = 0$ for null momenta

(12) $$\boxed{K_{\alpha\dot\alpha} = \lambda_\alpha(K)\tilde\lambda_{\dot\alpha}(K)}$$

pair of spinors associated to K_μ null

$(\text{if } K^2 \neq 0 \Rightarrow K_{\alpha\dot\alpha} = \sum_I \lambda^I_\alpha \tilde\lambda_{\dot\alpha I})$

Important Remarks:

- The $\lambda_\alpha(K)$ and $\tilde\lambda_{\dot\alpha}(K)$ solve Weyl equations:

(13) $$K_{\alpha\dot\alpha}\tilde\lambda^{\dot\alpha}(K) = 0 \qquad K^{\dot\alpha\alpha}\lambda_\alpha(K) = 0$$

by antisymmetry of $SL(2,\mathbb{C})$-invariant contractions, $\xi\lambda \equiv \xi^\alpha\lambda_\alpha = \xi_\beta\lambda_\alpha\epsilon^{\alpha\beta}$ $\tilde\xi\tilde\lambda = \tilde\xi_{\dot\alpha}\tilde\lambda^{\dot\alpha} = \tilde\xi^{\dot\beta}\tilde\lambda^{\dot\alpha}\varepsilon_{\dot\alpha\dot\beta}$ built out of $\varepsilon^{\alpha\beta}$ and $\varepsilon_{\dot\alpha\dot\beta}$

For instance, for $K^\mu = E(1,0,0,1) \Rightarrow \lambda_\alpha = \sqrt{2E}\begin{pmatrix}0\\1\end{pmatrix}_\alpha$ $\tilde\lambda_{\dot\alpha} = \sqrt{2E}\begin{pmatrix}0\\1\end{pmatrix}_{\dot\alpha}$ are ok

This immediately confirms λ_α is left-handed and $\tilde\lambda_{\dot\alpha}$ right-handed

(14) $$\begin{cases} \lambda_\alpha \xrightarrow{R_z(\theta)} \exp\left(-i\frac{\sigma^3}{2}\theta\right)_\alpha^{\ \beta}\lambda_\beta = \exp(i\theta/2)\lambda_\alpha \\ \tilde\lambda^{\dot\alpha} \longrightarrow \exp\left(-i\frac{\sigma^3}{2}\theta\right)_{\dot\alpha}^{\ \dot\beta}\tilde\lambda^{\dot\beta} = \exp(-i\theta/2)\tilde\lambda^{\dot\beta}\end{cases}$$

- For real momenta $\lambda^*_\alpha = \tilde\lambda_{\dot\alpha}$ since σ^μ are hermitian.

- The pair $(\lambda, \tilde\lambda)$ is defined from K^μ up to a rescaling

(15) $$\lambda_\alpha \to w\lambda_\alpha \qquad \tilde\lambda_{\dot\alpha} \to w^{-1}\tilde\lambda_{\dot\alpha} \qquad K^\mu \to K^\mu$$

That leaves K^μ invariant. (and for K_μ real $w^* = w^{-1} \Rightarrow w = \exp(i\theta/2)$ $\theta\in\mathbb{R}$)

This is nothing but the action of little group$_{k_\mu}$, $w = e^{+i\theta/2}$ in (14) example.

This means that $(\lambda, \tilde\lambda)$ are elements of equivalence class

(16) $\quad (\lambda, \tilde\lambda) \sim (\lambda', \tilde\lambda') \quad$ iff $\quad \lambda_\alpha \tilde\lambda_{\dot\alpha} = \lambda'_\alpha \tilde\lambda'_{\dot\alpha}$ $\quad$ ($= k_\mu$ the same)

We can choose a representative $(\lambda_\alpha(\bar k), \tilde\lambda_{\dot\alpha}(\bar k))$ for a certain $\bar k^\mu_{\alpha\dot\alpha} = \lambda_\alpha(\bar k)\tilde\lambda_{\dot\alpha}(\bar k)$

then define

(17)
$$\boxed{\begin{array}{ll} \lambda_\alpha(k) \equiv L_\alpha{}^\beta \lambda_\beta(\bar k) & L_\alpha{}^\beta \equiv L_\alpha{}^\beta(k,\bar k) \quad \tilde L_{\dot\alpha}{}^{\dot\beta} \equiv \tilde L_{\dot\alpha}{}^{\dot\beta}(k,\bar k) \text{ such that} \\ \lambda_{\dot\alpha}(k) \equiv \tilde L_{\dot\alpha}{}^{\dot\beta} \tilde\lambda_{\dot\beta}(\bar k) & L_\alpha{}^\beta \tilde L_{\dot\alpha}{}^{\dot\beta} \bar k_{\beta\dot\beta} = k_{\alpha\dot\alpha} \end{array}}$$

(i.e. $(L, \tilde L) : \bar k_{\alpha\dot\alpha} \to k_{\alpha\dot\alpha}$ and $L_\mu{}^\nu = L_\mu{}^\nu(L,\tilde L) : \bar k_\mu \to k_\mu$ via $SL(2,\mathbb{C})/\mathbb{Z}_2$)

moreover, $L = \tilde L$ if $k^\mu \in \mathbb{R}^4$)

Crucially, however, a generic $SL(2,\mathbb{C})$-tranformation change representative:

(18)
$$\Lambda_\alpha{}^\beta \lambda_\beta(k) = \Lambda_\alpha{}^\beta L_\beta{}^\gamma(k,\bar k)\, \lambda_\gamma(k) = L_\alpha{}^\beta(\Lambda k, \bar k) \underbrace{\left(L^{-1}(\Lambda k, k)\, \Lambda\, L(k,\bar k)\right)_\beta{}^\gamma}_{\text{rescale } \lambda \text{ by } w} \lambda_\gamma(\bar k)$$
$$\tilde\Lambda_{\dot\alpha}{}^{\dot\beta} \tilde\lambda_{\dot\beta}(k) = \dots = \tilde L_{\dot\alpha}{}^{\dot\beta}(\Lambda k, \bar k) \underbrace{\left(\tilde L^{-1}(\Lambda k, k)\, \tilde\Lambda\, \tilde L(k,\bar k)\right)_{\dot\beta}{}^{\dot\gamma}}_{\text{rescale } \tilde\lambda \text{ by } w^{-1}} \tilde\lambda_{\dot\gamma}(\bar k)$$

and
$$\left(L^{-1}(\Lambda k, k)\, \Lambda\, L\right)_\alpha{}^\beta \left(\tilde L^{-1}(\Lambda k, k)\, \tilde\Lambda\, \tilde L(k,\bar k)\right)_{\dot\alpha}{}^{\dot\beta} \bar k_{\beta\dot\beta} = L^{-1}(\Lambda k, \bar k)_\alpha{}^\beta \Lambda_\beta{}^\gamma\, \tilde L^{-1}(\Lambda k, \bar k)_{\dot\alpha}{}^{\dot\beta} \tilde\Lambda_{\dot\beta}{}^{\dot\gamma} k_{\gamma\dot\gamma}$$
$$= L^{-1}(\Lambda k, \bar k)_\alpha{}^\beta\, \tilde L^{-1}(\Lambda k, \bar k)_{\dot\alpha}{}^{\dot\beta} (\Lambda k)_{\beta\dot\beta} \quad (\leftarrow \sigma^\mu_{\beta\dot\beta} \Lambda_\mu{}^\nu k_\nu) \text{ via } SL(2,\mathbb{C})/\mathbb{Z}_2 \sim SO(3,1)$$
$$= \bar k_{\alpha\beta}$$

so that
$$\left\{ \begin{array}{l} \Lambda_\alpha{}^\beta \lambda_\beta(k) = L_\alpha{}^\beta(\Lambda k, \bar k)\, \lambda_\beta(\bar k)\, w = \lambda_\alpha(\Lambda k)\, w = \lambda_\alpha(\Lambda k)\, e^{+i\theta(\Lambda, k)} \\ \tilde\Lambda_{\dot\alpha}{}^{\dot\beta} \tilde\lambda_{\dot\beta}(k) = \tilde L_{\dot\alpha}{}^{\dot\beta}(\Lambda k, \bar k)\, \tilde\lambda_{\dot\beta}(\bar k)\, w^{-1} = \tilde\lambda_{\dot\alpha}(\Lambda k)\, w^{-1} = \tilde\lambda_{\dot\alpha}(\Lambda k)\, e^{-i\theta(\Lambda, k)} \end{array} \right.$$

Summerizing:

Given a null momentum (complex or real) we can associate to it a pair of "momentum" spinors $(\lambda_\alpha(k), \tilde\lambda_{\dot\alpha}(k))$ up to little-group transformations, and indeed $SL(2,\mathbb{C}) \times SL(2,\mathbb{C})$ change representative in equiv. class:

$$(19) \qquad \boxed{\Lambda_\alpha{}^\beta \lambda_\beta(k) = \underbrace{e^{\frac{i\theta(\Lambda,k)}{2}}}_{= w} \lambda_\alpha(\Lambda k) \qquad \tilde\Lambda_{\dot\alpha}{}^{\dot\beta} \tilde\lambda_{\dot\beta}(k) = \underbrace{e^{-\frac{i\theta(\Lambda,k)}{2}}}_{= w^{-1}} \tilde\lambda_{\dot\alpha}(\Lambda k)}$$

This is important because it allows to solve the constraint (8), namely

$$(20) \qquad \boxed{M_{h_1\dots h_n}(\Lambda k_1, \dots, \Lambda k_n) = w_1^{2h_1} \cdots w_n^{2h_n} M_{h_1 \dots h_n}(k_1, \dots k_n)} \qquad w_i = e^{i\frac{\theta(\Lambda,k_i)}{2}} = e^{\frac{i\theta_i}{2}}$$

as soon as $M_{h_1\dots h_n}$ is actually a momentum-spinor $SL(2,\mathbb{C})\times SL(2,\mathbb{C})$-invariant function $M_{h_1\dots h_n}(\lambda^i, \tilde\lambda^i)$ (with $\lambda^i \equiv \lambda(k_i)$, $\tilde\lambda^i \equiv \tilde\lambda(k_i)$)

$$(20) \qquad M_{h_1\dots h_n}(\lambda^i, \tilde\lambda^i) = \hat{M}_{h_1\dots h_n}(\lambda^{i\alpha}\lambda^j_\alpha, \tilde\lambda^i_{\dot\alpha}\tilde\lambda^{j\dot\alpha}) \qquad SL(2,\mathbb{C})_{\mathbb{C}} \text{ invariant}$$

of homogeneous degree $2h_i$ for each i-th particle

$$(21) \qquad \boxed{M_{h_1\dots h_n}(w_i^{-1}\lambda^i, w_i \tilde\lambda^i) = w_i^{2h_i} M_{h_1\dots h_n}(\lambda^i, \tilde\lambda^i)} \; \forall_i \qquad \text{Homogeneous w.r.t. Little group.}$$

Indeed $M_{h_1\dots h_n}(\Lambda k_1 \dots \Lambda k_n) = M_{h_1\dots h_n}(\lambda(\Lambda k_i), \tilde\lambda(\Lambda k_i)) \underset{(19)}{=} M_{h_1 \dots h_n}(w_i^{-1}(\Lambda\lambda)^i, w_i(\tilde\Lambda\tilde\lambda)^i)$

$$\underset{(20)}{=} M_{h_1\dots h_n}(w^{-1}\lambda^i, w_i\tilde\lambda^i)$$

$$\underset{(21)}{=} \left(\prod_i w_i^{+2h_i}\right) M_{h_1\dots h_n}(\lambda^i, \tilde\lambda^i) = \prod_i e^{+ih_i\theta_i(\Lambda,k)} M_{h_1\dots h_n}(k_1 \dots k_n)$$

Remark: The condition (21) says amplitudes are $SL(2,\mathbb{C})_{\mathbb{C}}$-invariant solutions of a diff. problem:

$$(22) \qquad \boxed{\mathbb{H}_i M_{h_1\dots h_n} = h_i M_{h_1\dots h_n} \qquad \mathbb{H}_i \equiv \frac{1}{2}\left(\tilde\lambda^i_{\dot\alpha}\frac{\partial}{\partial\tilde\lambda^i_{\dot\alpha}} - \lambda^i_\alpha \frac{\partial}{\partial \lambda^i_\alpha}\right) \qquad \forall i}$$

where the helicity operator $\mathbb{H}_i$ counts (half right − half left) spinors in $M_{h_1 \dots h_n}$.
The constraint (22) is not enough, amplitudes obey unitarity too: so the strategy is building 3pt obeying (22), i.e. (21), and construct 4pt that automatically will satisfy unitarity.

Updated lesson: Amplitudes are $SL(2,\mathbb{C})_\alpha$-invariant functions of momentum spinors (as opposed to just momenta) that obey (22) i.e. solve (21)

— Angle- and square-spinors —

It's convenient to introduce a index-free notation:

$$\begin{array}{lll} \lambda^i_\alpha = |i\rangle & \tilde\lambda^i_{\dot\alpha} = [i| & \lambda^i_\alpha \tilde\lambda^i_{\dot\alpha} = |i\rangle[i| = K^i_{\alpha\dot\alpha} \\ \lambda^{i\alpha} = \langle i| & \tilde\lambda^{i\dot\alpha} = |i] & \tilde\lambda^{i\dot\alpha}\lambda^{i\alpha} = |i]\langle i| = K^{i\dot\alpha\alpha} \\ \lambda^{j\alpha}\lambda^i_\alpha = \langle j\, i\rangle & \tilde\lambda^i_{\dot\alpha}\tilde\lambda^{j\dot\alpha} = [i\, j] & \langle i\, i\rangle = 0 = [i\, i] \end{array} \tag{23}$$

$$\langle i|K|j] = \langle i\, K\rangle[K\, j]$$

$$(K_i + K_j)^2 = s_{ij} = \cancel{K_i^2} + \cancel{K_j^2} + 2K_i\cdot K_j = \langle i\, j\rangle[j\, i] = \mathrm{Tr}(K^j K^i) = K^j_\mu K^i_\nu \mathrm{Tr}(\sigma^\mu\bar\sigma^\nu)$$

$$\Sigma_i |i\rangle[i| = 0 = \Sigma_i K^i_{\alpha\dot\alpha} \quad \text{(mom. conservation)}$$

so that amplitudes built with $\langle i\, j\rangle$ and $[i\, j]$

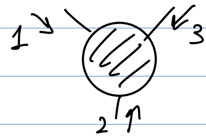
$$\underline{M_{h_1\dots h_n}(\langle i\, j\rangle, [i\, j])} \tag{24}$$

obey Lorentz as soon as are $2h_i$-homogeneous w.r.t. Little group$_i$

$$\boxed{M_{h_1\dots h_n}(\langle i\, j\rangle\, w_i^{-1}, w_i[i\, j]) = w_i^{2h_i} M_{h_1\dots h_n}(\langle i j\rangle, [ij]) \quad \forall i} \tag{25}$$

Let's find a solution of (25) for 3pt functions.

— Massless 3pt-functions —



3pt-kinematics is very simple:

(26) $$K_1+K_2+K_3=0 \Rightarrow (K_i+K_j)^2 \equiv s_{ij} \underset{\text{mom. cons.}}{=} m^2_{k\neq i,j}$$

so that all Mandelstam invariants are constant masses. For all massless particles the Mandelstam invariants are all trivial:

(27) $$2K_i\cdot K_j = 0 = \langle i j\rangle[j i] \Rightarrow \text{either } \langle ij\rangle = 0 \text{ or } [j i]=0$$

Example: $\langle 12\rangle[21]=0 \Rightarrow$ say $\langle 12\rangle\neq 0$ $[12]=0 \Rightarrow \langle 12\rangle[23] \underset{\text{mom. cons.}}{=} -\langle 11\rangle[13]-\langle 13\rangle[33]$

$\Rightarrow [23]=0$ and $\langle 23\rangle[31] \underset{\text{mom. cons.}}{=} 0 \Rightarrow [31]=0$
$\langle 23\rangle\neq 0$

Lesson: (3pt massless) if one $[ij]=0$ then all $[ij]=0$; if instead one $\langle ij\rangle=0$ then all $\langle ij\rangle=0$

Remark:

For real kinematics, actually, both are vanishing since $|i\rangle^*=[i| \Rightarrow [ij]=\langle ji\rangle^*=0$ and therefore no non-trivial on-shell 3pt-amplitude exist for real momenta.
This is way it's useful to go to complex kinematics

This immediately imply that 3pt-functions are either functions of $\langle,\rangle$ or $[,]$, not both:

(28) $$\boxed{\mathcal{M}_{h_1h_2h_3}(\langle ij\rangle) \quad \text{or} \quad \mathcal{M}_{h_1h_2h_3}([i,j])}$$

called "holomorphic" configuration — "anti-holomorphic" configuration

Now, we need to enforce the homogeneity-helicity constraint (25) $\forall i$:

If holomorphic:

(29) $$M_{h_1 h_2 h_3} = c_{a_1 a_2 a_3} \langle 12\rangle^{a_1} \langle 23\rangle^{a_2} \langle 31\rangle^{a_3} \longrightarrow \begin{cases} -a_1 - a_3 = 2h_1 \\ -a_1 - a_2 = 2h_2 \\ -a_3 - a_2 = 2h_3 \end{cases}$$ solve for a_i

(30) $$a_1 = -h_1 - h_2 + h_3 \qquad a_2 = h_1 - h_2 - h_3 \qquad a_3 = h_2 - h_1 - h_3$$

$\Downarrow$

(31) $$\boxed{M_{h_1 h_2 h_3} = \kappa \cdot \langle 12\rangle^{h_3 - h_1 - h_2} \langle 23\rangle^{h_1 - h_2 - h_3} \langle 31\rangle^{h_2 - h_1 - h_3}}$$

then if anti-holomorphic instead

(32) $$\boxed{M_{h_1 h_2 h_3} = \bar{\kappa}\, [12]^{-h_3 + h_1 + h_2} [23]^{-h_1 + h_2 + h_3} [31]^{-h_2 + h_1 + h_3}}$$

How to choose if holomorphic or antiholomorphic? Notice that $\langle ij\rangle \sim \text{Energy} \sim [ij]$

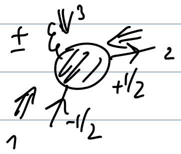
(33) $$M_{h_1 h_2 h_3} \sim \begin{cases} (\text{Energy})^{-(h_1 + h_2 + h_3)} & \text{holomorphic.} \\ (\text{Energy})^{+(h_1 + h_2 + h_3)} & \text{anti-holomorphic.} \end{cases}$$

$\Downarrow$

Locality, in the form of $(\text{Energy})^{\text{positive power}}$, select which one to use depending on whethe $\sum_i h_i > 0$ or $\sum h_i < 0$.

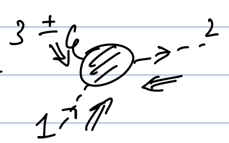
(This condition of locality can be understood by matching it to Lagrangian, with fields and ∂^μ, no inverse-derivatives)

Example: Minimal coupling massless left-handed fermion to photon: ("helicity preserving")

(34) $M_{\frac{1}{2}\, -\frac{1}{2}\, \mp 1} =$ 

$$= \begin{cases} g \dfrac{\langle 23\rangle^2}{\langle 12\rangle} & h_3 = - \\ g \dfrac{[31]^2}{[12]} & h_3 = + \end{cases}$$

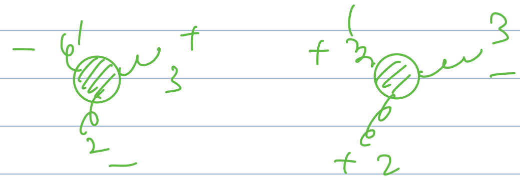
Example: Minimal coupling massless charged scalar to to photon

(35) $M_{0\,0\,\mp} =$ 

$$= \begin{cases} g \dfrac{\langle 23\rangle\langle 31\rangle}{\langle 12\rangle} & h_3 = - \\ g \dfrac{[23][31]}{[12]} & h_3 = + \end{cases}$$

(Notice that (35) is antisym. under $1 \leftrightarrow 2$ $\Rightarrow$ 1 & 2 can't be identical, you need indeed two states of opposite charge.)

Example: YM



(36) $M_{--+} \propto \dfrac{\langle 12\rangle^3}{\langle 13\rangle\langle 32\rangle}$ $\qquad M_{++-} \propto \dfrac{[12]^3}{[13][32]}$

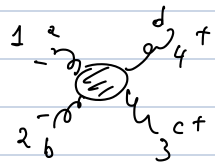
Notice that antisym under $1 \leftrightarrow 2$ $\Rightarrow$ the coupling constant must be antisym.

(37) $$\boxed{M(1_a^- 2_b^- 3_c^+) = g f_{abc} \frac{\langle 12\rangle^3}{\langle 13\rangle\langle 32\rangle} \qquad M(1_a^+ 2_b^+ 3_c^-) = g f_{abc} \frac{[12]^3}{[13][32]}}$$

— 4pt-Amplitudes —

The strategy here is the following:
- Make ansatz compatible w./ Little Group
- Solve for unknown parameters using factorizat.

Let's consider the instructive case of YM amplitude:

(38) $M(1_a^- 2_b^- 3_c^+ 4_d^+) =$  $= \langle 12\rangle^2 [34]^2 F(s,t,u)$

(other ansatz such as $\frac{\langle 12\rangle^2}{\langle 34\rangle^2}\tilde{F} = \frac{\langle 12\rangle^2 [34]^2}{\langle 34\rangle^2 [43]^2}\tilde{F} = \langle 12\rangle^2 [34]^2 \frac{\tilde{F}(s,t,u)}{s}$ included)

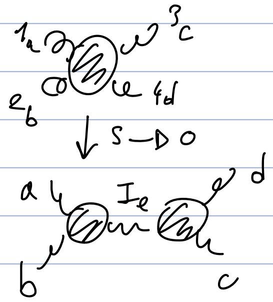
Then we assume we are working at tree-level and have a dimensionless gauge coupling: the most general F with simple poles in the 3-channels is

(39) $$M(1_a^- 2_b^- 3_c^+ 4_d^+) = \langle 12\rangle^2 [34]^2 \left(\frac{c_{st}}{st} + \frac{c_{tu}}{tu} + \frac{c_{us}}{us} \right)$$

(analytic terms can be added, but they come with $1/\Lambda^n$ scale, corresponding to higher-dim. operators than YM-theory)

Now, we demand that on each pole the amplitude factorize in 3pt's:

(40) $$\operatorname*{Res}_{s=0} M(1_a^- 2_b^- 3_c^+ 4_d^+) = \langle 12\rangle^2 [34]^2 \frac{1}{t}\left(c_{st} - c_{us}\right) = -M(1_a^- 2_b^- I_e^+)\, M(3_c^+ 4_d^+ I_e^-)$$



$$\overset{(37)}{=} -g^2 f_{abe} f_{cde} \frac{\langle 12\rangle^3}{\langle 1I\rangle\langle I2\rangle}\frac{[34]^3}{[3I][I4]}$$

$$= -g^2 f_{abe} f_{cde} \frac{\left(\langle 12\rangle [34]\right)^3}{\langle 1I\rangle [I3]\, \langle 2I\rangle [I4]}$$

$$\underset{\text{mom. conservation}}{=} -g^2 f_{abe} f_{cde} \frac{\left(\langle 12\rangle [34]\right)^3}{\langle 12\rangle [23]\langle 23\rangle [34]} = +g^2 f_{abe} f_{cde} \frac{\langle 12\rangle^2 [34]^2}{u}$$

mom. conservation

$$|I\rangle [I| = -|1\rangle[1| - |2\rangle[2| = -|3\rangle[3| - |4\rangle[4|$$

$$= -g^2 f_{abe} f_{cde} \frac{\langle 12\rangle^2 [34]^2}{t} \qquad \text{on } s \to 0 \text{ limit}$$

$$\Downarrow$$

(41) $$\begin{cases} c_{st} - c_{us} = -g^2 f_{abe} f_{cde} \\ c_{tu} - c_{st} = -g^2 f_{bce} f_{ade} \\ c_{us} - c_{tu} = -g^2 f_{cae} f_{bde} \end{cases}$$

& Repeating for other channels $t \to 0$ and $u \to 0$

$$0 = f_{abe} f_{cde} + f_{bce} f_{ade} + f_{cae} f_{bde}$$ Jacobi identity!

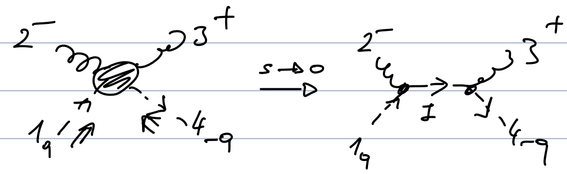
Under this condition on the f's, the amplitude is fully determined.

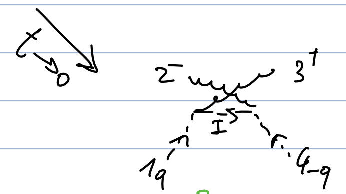
(42) $$\boxed{M(1_a^- 2_b^- 3_c^+ 4_d^+) \underset{s+t+u=0\,+\,(41)}{=} -g^2 \langle 12\rangle^2 [34]^2 \left(\frac{f_{abe} f_{cde}}{st} + \left(f_{bce} f_{ade} + f_{abe} f_{cde}\right)\frac{1}{tu}\right)}$$

and by construction consistent with unitarity/factorization.

Let's see another:

Example: Compton scattering for charged scalars





(43) $$M(1_q^0\, 2^-\, 3^+\, 4_{-q}^0) = g^2 \langle 21\rangle^2 [13]^2 \underbrace{F(s,t,u)}_{\left(\frac{c_{st}}{st} + \frac{c_{su}}{su} + \frac{c_{tu}}{tu}\right)}$$

[[Equivalent and more symm. w.r.t. 1 & 4 is $g^2 \langle 2|p_1|3]^2 = \langle 2|p_1 - p_4|3]^2 \frac{g^2}{4}$

since $p_4 = |4\rangle[4| \overset{=}{\underset{\text{mom. cons.}}{}} -|1\rangle[1| - |2\rangle[2| - |3\rangle[3|$.

Yet another equivalent is $\langle 12\rangle\langle 24\rangle[43][31] \underset{\text{mom. cons.}}{=} \langle 21\rangle^2[13]^2$]]

s-channel factorization:

$\left(M_{00\mp} = \right.$  $\left. = \begin{cases} g\,\frac{\langle 23\rangle\langle 31\rangle}{\langle 12\rangle} & h_3 = - \\ g\,\frac{[23][31]}{[12]} & h_3 = + \end{cases} \right)$

(44) $$M(1_q^0\, 2^-\, 3^+\, 4_{-q}^0) \xrightarrow[s\to 0]{(43)} g^2 \frac{\langle 21\rangle^2 [13]^2}{s} \left(\frac{c_{st} - c_{su}}{t}\right)$$

$\downarrow s\to 0$

$$-M(1_q^0\, 2^-\, I_{-q}^0)\frac{1}{s} M(I_{+q}^0\, 3^+\, 4_{-q}^0) \underset{(35)}{=} -g^2 \frac{\langle I2\rangle\langle 21\rangle}{\langle 1I\rangle}\frac{1}{s}\frac{[43][3I]}{[I4]} \underset{\text{mom. cons.}}{=} -g^2 \frac{\langle 21\rangle^2 [43][13]}{\langle 13\rangle[34]}\frac{1}{s}\frac{[31]}{[31]}$$

$$= -g^2 \frac{\langle 21\rangle^2 [13]^2}{st}$$

(45) $$c_{st} - c_{su} = -1 \quad \Leftarrow$$

t-channel factorization

$$M(1_q^0\, 2^-\, 3^+\, 4_{-q}^0) \xrightarrow[t\to 0]{(43)} g^2 \frac{\langle 21\rangle^2[13]^2}{t} \left(\frac{c_{st} - c_{tu}}{s}\right)$$

$\downarrow$

$$-M(1_q^0\, 3^+\, I_{-q}^0)\frac{1}{t} M(I_q^0\, 2^-\, 4_{-q}^0) = -g^2 \frac{[I3][31]}{[1I]}\frac{1}{t}\frac{\langle 42\rangle\langle 2I\rangle}{\langle I4\rangle} \underset{\text{mom. cons.}}{=} +\frac{g^2}{t}\frac{[31]^2\langle 42\rangle\langle 21\rangle}{\langle 42\rangle[21]}\frac{\langle 12\rangle}{\langle 12\rangle}$$

$$= -g^2 \frac{\langle 21\rangle^2[31]^2}{ts}$$

(46) $$c_{st} - c_{tu} = -1 \quad (\&\ c_{tu} = c_{su}) \quad \Leftarrow$$

$\Downarrow$

(47) $$M(1_q^0\, 2^-\, 3^+\, 4_{-q}^0) = g^2\langle 21\rangle^2[13]^2\left(\frac{c_{st}}{st} + \frac{c_{su}}{su} + \frac{c_{tu}}{tu}\right) \underset{(46)+(45)}{=} -g^2\frac{\langle 21\rangle^2[13]^2}{st}$$

(48)

$$\left(\left(\frac{1}{st} + \frac{1}{su} + \frac{1}{tu}\right) = 0\right)$$